Quantum nonlocality test for continuous-variable states with dichotomic observables

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There have been theoretical and experimental studies on quantum nonlocality for continuous variables, based on dichotomic observables. In particular, we are interested in two cases of dichotomic observables for the light field of continuous variables: One case is even and odd numbers of photons and the other case is no photon and the presence of photons. We analyze various observables to give the maximum violation of Bell's inequalities for continuous-variable states. We discuss an observable which gives the violation of Bell's inequality for any entangled pure continuous-variable state. However, it does not have to be a maximally entangled state to give the maximal violation of Bell's inequality. This is attributed to a generic problem of testing the quantum nonlocality of an infinite-dimensional state using a dichotomic observable.

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I. INTRODUCTION

The paradox suggested by Einstein, Podolsky, and Rosen aroused controversy about nonlocality of quantum states [1]. Bell proposed a remarkable inequality imposed by a local hidden variable theory [2], which enables a quantitative test on quantum nonlocality. Numerous theoretical studies and experimental demonstrations have been performed to understand nonlocal properties of quantum states. Various versions of Bell's inequality [3,4] followed the original one [2].

Gisin and Peres found pairs of observables whose correlations violate Bell's inequality for a discrete N-dimensional entangled state [5]. Banaszek and Wódkiewicz (BW) studied Bell's inequality for continuous-variable states, in terms of Wigner representation in phase space based upon parity measurement and displacement operation [6]. This is useful because of its experimental relevance, but does not lead to maximal violation for the original Einstein-Podolsky-Rosen (EPR) state [7]. Recently, Chen *et al.* studied Bell's inequality of continuous-variable states [8] using their newly defined Bell operator [8,9]. In contrast to the operators in BW formalism, the pseudospin operators are not experimentally easy to realize, but the EPR state can maximally violate Bell's inequality in their framework [8].

In this paper, we relate the "pseudospin" Bell operator of Chen *et al.* to one of Gisin and Peres for a finite-dimensional state to bridge the gap between the discussions for the nonlocality of finite- and infinite-dimensional (or continuous-variable) systems. The origin of the pseudospin operator is attributed to the limiting case of Gisin-Peres observable [5]. We investigate various versions of Clauser, Horne, Shimony, and Holt's (CHSH's) inequality for continuous-variable states. It is pointed out that the BW formalism can be generalized to obtain a larger Bell violation [10], but it cannot give the maximal violation for the EPR state even in the generalized version. We analyze the reason why the EPR state cannot maximally violate Bell's inequality in the generalized BW formalism. We compare the EPR state with an entangled state of two coherent states [11]. In contrast to the EPR state, the entangled coherent state shows the maximal Bell violation for certain limit both for the generalized BW and *et al.* formalism of for the Chen. We also investigate Clauser and Horne's (CH) version of Bell's inequality. We find the upper and lower bounds for the Bell-CH inequality and test whether the values for continuous-variable states reach these bounds.

II. ORIGIN OF PSEUDOSPIN OPERATOR

Chen *et al.*, introduced a pseudospin operator $\mathbf{s}=(s_x,s_y,s_z)$ for a nonlocality test of continuous variables as a direct analogy of a spin-1/2 system [8,9],

$$s_z=\sum_{n=0}^{\infty}(|2n+1\rangle\langle 2n+1|-|2n\rangle\langle 2n|), \tag{1}$$

$$s_x\pm s_y=2s_{\pm}, \tag{2}$$

$$\mathbf{a}\cdot\mathbf{s}=s_z\cos\theta+\sin\theta(e^{i\varphi}s_-+e^{-i\varphi}s_+), \tag{3}$$

where $s_-=\Sigma_{n=0}^{\infty}|2n\rangle\langle 2n+1|=(s_+)^{\dagger}$ and $\mathbf{a}$ is a unit vector. The Bell-CHSH operator based upon the pseudospin operator is then defined as [3,8]

$$\mathcal{B}=(\mathbf{a}\cdot\mathbf{s}_1)\otimes(\mathbf{b}\cdot\mathbf{s}_2)+(\mathbf{a}\cdot\mathbf{s}_1)\otimes(\mathbf{b}'\cdot\mathbf{s}_2)+(\mathbf{a}'\cdot\mathbf{s}_1)\otimes(\mathbf{b}\cdot\mathbf{s}_2)-(\mathbf{a}'\cdot\mathbf{s}_1)\otimes(\mathbf{b}'\cdot\mathbf{s}_2), \tag{4}$$

where 1 and 2 denote two different modes and $\mathbf{a}'$, $\mathbf{b}$, and $\mathbf{b}'$ are unit vectors.

Bell's inequality imposed by local hidden variable theory is then $|\langle\mathcal{B}\rangle|\leqslant 2$. In this formalism, the violation of the inequality is limited by Cirel'son bound $|\langle\mathcal{B}\rangle|\leqslant 2\sqrt{2}$ [8,12]. It was found that a two-mode squeezed state

$$|\mathrm{TMSS}\rangle=\sum_{n=0}^{\infty}\frac{(\tanh r)^n}{\cosh r}|n\rangle|n\rangle, \tag{5}$$

where $|n\rangle$ is a number state and r is the squeezing parameter, maximally violates Bell's inequality, i.e., $|\langle\mathcal{B}\rangle|_{max}\to 2\sqrt{2}$

when r becomes infinity [8]. Note that the two-mode squeezed state (5) becomes the original EPR state when $r \to \infty$ [13].

Gisin and Peres found pairs of observables whose correlations violate Bell's inequality for an N-dimensional entangled state [5]

$$|\Psi\rangle = \sum_{n=0}^{N-1} c_n |\phi_n\rangle |\psi_n\rangle, \tag{6}$$

where $\{|\phi_n\rangle\}$ and $\{|\psi_n\rangle\}$ are any orthonormal bases. Further they showed that the violation of Bell's inequality is maximal in the case of a spin-j singlet state for an j even. The Gisin-Peres observable is

$$A(\theta) = \Gamma_x \sin\theta + \Gamma_z \cos\theta + \mathcal{E}, \tag{7}$$

where Γ_x and Γ_z are block-diagonal matrices in which each block is an ordinary Pauli matrix, σ_x and σ_z, respectively. $\mathcal{E}$ is a matrix whose only nonvanishing element is $\mathcal{E}_{N-1,N-1} = 1$ when N is odd and $\mathcal{E}$ is zero when N is even. The Bell operator is then defined as

$$\begin{aligned}\mathcal{B}_{GP} = &(\mathbf{a}\cdot A_1)\otimes(\mathbf{b}\cdot A_2) + (\mathbf{a}\cdot A_1)\otimes(\mathbf{b}'\cdot A_2) + (\mathbf{a}'\cdot A_1)\\ &\otimes(\mathbf{b}\cdot A_2) - (\mathbf{a}'\cdot A_1)\otimes(\mathbf{b}'\cdot A_2),\end{aligned} \tag{8}$$

where A represents the Gisin-Peres observable $A(\theta)$. It was Gisin [14] who showed any entangled pure state violates a Bell's inequality. Later, Gisin and Peres [5] found the observable (7) to give the violation of Bell's inequality for any N-dimensional entangled pure state.

In limit $N \to \infty$, we find that Γ_x and Γ_z become pseudospin operators s_x and s_z in Eq. (2), and $A(\theta)$ becomes $\mathbf{a}\cdot\mathbf{s}$ (with $\varphi = 0$) in Eq. (3). Note that the effect of $\mathcal{E}$ vanishes for $N \to \infty$. Understanding the observables of Chen *et al.* as a limiting case of Gisin-Peres observable defined for a finite discrete system, it is now straightforward to show that the EPR state maximally violates Bell's inequality as the EPR state $\Sigma_{n=0}^{\infty}|n\rangle|n\rangle$ is the *infinite-dimensional singlet state*. Extending the Gisin and Peres' argument, we can make a remark: Any bipartite pure infinite-dimensional entangled state violates Bell's inequality for observables based on the pseudospin observables.

III. THE BELL-CHSH INEQUALITIES FOR CONTINUOUS VARIABLES

A. The two-mode squeezed state

Banaszek and Wódkiewicz studied Bell's inequality for continuous-variable systems based upon parity measurement and displacement operation [6]:

$$\begin{aligned}\Pi(\alpha) &= \Pi^{+}(\alpha) - \Pi^{-}(\alpha)\\ &= D(\alpha)\sum_{n=0}^{\infty}(|2n\rangle\langle 2n| - |2n+1\rangle\langle 2n+1|)D^{\dagger}(\alpha),\end{aligned} \tag{9}$$

where $D(\alpha)$ is the displacement operator $D(\alpha) = \exp[\alpha\hat{a}^{\dagger} - \alpha^{*}\hat{a}]$ for bosonic operators $\hat{a}$ and $\hat{a}^{\dagger}$. It should be pointed out that in order to maximize the violation of Bell's inequality, the BW formalism needs to be generalized to write the Bell operator as [10]

$$\begin{aligned}\mathcal{B}_{BW} = &\Pi_1(\alpha)\Pi_2(\beta) + \Pi_1(\alpha')\Pi_2(\beta) + \Pi_1(\alpha)\Pi_2(\beta')\\ &- \Pi_1(\alpha')\Pi_2(\beta').\end{aligned} \tag{10}$$

BW assumed two of the four parameters equal to zero as $\alpha = \beta = 0$. The Bell-CHSH inequality can then be represented by the Wigner function as

$$\begin{aligned}|\langle\mathcal{B}_{BW}\rangle| = &\frac{\pi^2}{4}|W(\alpha,\beta) + W(\alpha,\beta') + W(\alpha',\beta) - W(\alpha',\beta')|\\ &\leqslant 2,\end{aligned} \tag{11}$$

where $W(\alpha,\beta)$ represents the Wigner function of a given state. Using $\Pi_1(\alpha)\Pi_1(\alpha) = \Pi_2(\alpha)\Pi_2(\alpha) = \mathbb{1}$, it is straightforward to check the Cirel'son bound $|\langle\mathcal{B}_{BW}\rangle| \leqslant 2\sqrt{2}$ in the generalized BW formalism.

The Wigner function of the two-mode squeezed state is [15]

$$\begin{aligned}W_{TMSS}(\alpha,\beta) = &\frac{4}{\pi^2}\exp[-2\cosh 2r(|\alpha|^2 + |\beta|^2)\\ &+ 2\sinh 2r(\alpha\beta + \alpha^{*}\beta^{*})],\end{aligned} \tag{12}$$

with which the Bell function $B_{BW} \equiv \langle\mathcal{B}_{BW}\rangle$ can be calculated. In the infinite squeezing limit, the absolute Bell function maximizes as $|B_{BW}|_{max} \to 8/\sqrt[8]{3^9} \simeq 2.32$ at $\alpha = -\alpha' = \beta'/2 = \sqrt{(\ln 3)/16\cosh 2r}$ and $\beta = 0$. This shows that the EPR state does not maximally violate Bell's inequality in the generalized BW formalism. In Fig. 1(a), using the generalized BW formalism, the maximized value $|B_{BW}|_{max}$ is plotted for the two-mode squeezed state and compared with the violation of Bell's inequality based on other formalisms. (The method of steepest descent [16] is used in Fig. 1(a) and other figures in the paper to get the maximized value of violation within the formalism.)

The reason why the generalized BW formalism does not give the maximum violation for the EPR state can be explained as follows. The operator s_z in Eq. (1) is equivalent to BW's observable $\Pi(\alpha)$ when $\alpha = 0$ except a trivial sign change. The main difference is that BW use the displacement operator while Chen *et al.* use the direct analogy of the rotation of spin operators. When the Gisin-Peres observable $A(\theta)$ (or equivalently pseudospin observable $\mathbf{a}\cdot\mathbf{s}$ with $\varphi = 0$) is applied on an arbitrary state $\Sigma_{n=0}^{\infty} f(n)|n\rangle$, where $f(n)$ is an arbitrary function, we obtain

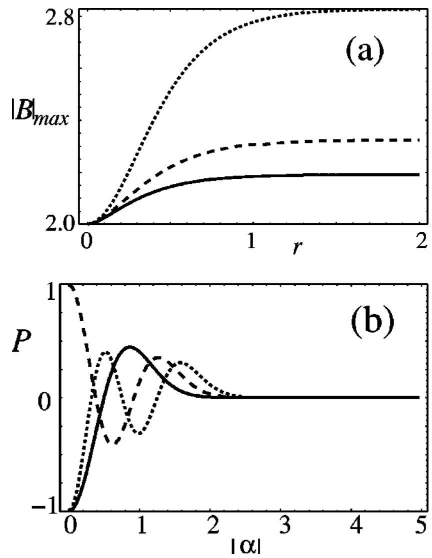

FIG. 1. (a) The maximized value of an absolute Bell function $|B|_{max}$ for a two-mode squeezed state vs the squeezing parameter r in the BW (solid line), the generalized BW (dashed), and the formalisms of Chen *et al.* (dotted). It is shown that the EPR state does not maximally violate Bell's inequality in the generalized BW formalism. (b) The expectation value P of BW's observable for number states of $n=1$ (solid), $n=2$ (dashed), and $n=3$ (dotted) is plotted against the absolute displacement parameter $|\alpha|$.

$$A(\theta)\sum_{n=0}^{\infty} f(n)|n\rangle=\sqrt{2}\cos(\theta-\pi/4)\sum_{n=0}^{\infty} f(2n)|2n\rangle + \sqrt{2}\sin(\theta-\pi/4)\sum_{n=0}^{\infty} f(2n+1)|2n+1\rangle. \tag{13}$$

The operator $A(\theta)$ rotates $\Sigma f(n)|n\rangle$ into even and odd parity states; the pseudospin observable (3) can completely flip the parity of any given state by changing the angle. Note that the only measurement applied to the nonlocality test here is the parity measurement. Different from the pseudospin operator, BW's observable $\Pi(\alpha)$ does not assure the complete parity change, which makes it impossible to find the maximal Bell violation of the two-mode squeezed state. In the two-mode squeezed state, orthogonal number states, which have well-defined parity, are the entangled elements. The expectation value of BW's observable for a number state is obtained as [17]

$$P(n,|\alpha|)=\langle n|\Pi(\alpha)|n\rangle = \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}\sum_{k=0}^{\infty}\left\{\frac{(2k)!}{|\alpha|^{4k}}[L_{2k}^{(n-2k)}(|\alpha|^2)]^2 - \frac{(2k+1)!}{|\alpha|^{4k+2}}[L_{2k+2}^{(n-2k-1)}(|\alpha|^2)]^2\right\}, \tag{14}$$

where $L_q^{(p)}(x)$ is an associated Laguerre polynomial. We numerically assess $P(n,|\alpha|)$ for some different numbers and check that the parity of the number states cannot be perfectly flipped by changing the parameter α of the displacement operator $D(\alpha)$ as shown in Fig. 1(b).

B. The entangled coherent state

The entangled coherent state [11] is an another important continuous-variable entangled state. Many possible applications to quantum information processing have been studied utilizing entangled coherent states [18]. The entangled coherent state $|\mathrm{ECS}\rangle$ can be defined as

$$|\mathrm{ECS}\rangle=\mathcal{N}(|\gamma\rangle|-\gamma\rangle-|-\gamma\rangle|\gamma\rangle), \tag{15}$$

$$|\gamma\rangle=e^{-|\gamma|^2/2}\sum_{n=0}^{\infty}\frac{\gamma^n}{\sqrt{n!}}|n\rangle, \tag{16}$$

where $\mathcal{N}$ is a normalization factor and $|\gamma\rangle$ is a coherent state with $\gamma\neq 0$. For the case of the entangled coherent state, the Bell function in the generalized BW formalism (11) can be calculated from its Wigner function

$$\begin{aligned}W_{ECS}(\alpha,\beta)=4\mathcal{N}^2\{&\exp[-2|\alpha-\gamma|^2-2|\beta+\gamma|^2]\\ &+\exp[-2|\alpha+\gamma|^2-2|\beta-\gamma|^2]\\ &-\exp[-2(\alpha-\gamma)(\alpha^*+\gamma)-2(\beta+\gamma)\\ &\times(\beta^*-\gamma)-4\gamma^2]-\exp[-2(\alpha^*-\gamma)(\alpha+\gamma)\\ &-2(\beta^*+\gamma)(\beta-\gamma)-4\gamma^2]\},\end{aligned} \tag{17}$$

where γ is assumed to be real for simplicity. We find that the Bell function approaches to $2\sqrt{2}$ for $\gamma\to\infty$ [10] at $\alpha=0$, $\beta=5\pi/16\gamma$, $\alpha'=\pi/8\gamma$, and $\beta'=3\pi/16\gamma$ as shown in Fig. 2(a).

The entangled coherent state can be represented in the (2×2)-Hilbert space as

$$|\mathrm{ECS}\rangle=\frac{1}{\sqrt{2}}(|e\rangle|d\rangle-|d\rangle|e\rangle), \tag{18}$$

where $|e\rangle=\mathcal{N}_+(|\gamma\rangle+|-\gamma\rangle)$ and $|d\rangle=\mathcal{N}_-(|\gamma\rangle-|-\gamma\rangle)$ are even and odd macroscopic quantum interference states with normalization factors $\mathcal{N}_+$ and $\mathcal{N}_-$. Note that these states form an orthogonal basis, regardless of the value of γ, which span the two-dimensional Hilbert space. Suppose that an ideal rotation $R_x(\theta)$ around the x axis,

$$\begin{aligned}R_x(\theta)|e\rangle&=\cos\theta|e\rangle+i\sin\theta|d\rangle,\\ R_x(\theta)|d\rangle&=i\sin\theta|e\rangle+\cos\theta|d\rangle\end{aligned} \tag{19}$$

can be performed on both sides of the entangled coherent state (18). Because, state (18) is the same as the EPR-Bohm state of a two-qubit system, it can be easily proved that it maximally violates the Bell's inequality, i.e., the maximized Bell function is $2\sqrt{2}$. Remarkably, it is known that the displacement operator acts like the rotation $R_x(\theta)$ on the even and odd macroscopic quantum interference states for $\gamma\gg 1$ [10,19]. The fidelity can be checked that

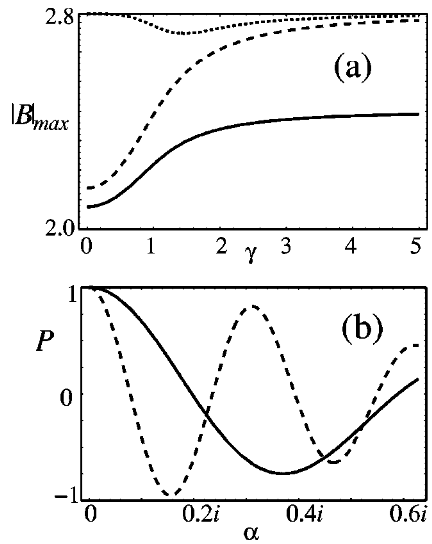

FIG. 2. (a) The maximized value of an absolute Bell function $|B|_{max}$ for an entangled coherent state is plotted against its coherent amplitude γ using the BW (solid), the generalized BW (dashed), and formalisms of Chen *et al.* (dotted). The entangled coherent state maximally violates Bell's inequality in the generalized BW formalism for $\gamma\to\infty$ and in the formalism of Chen *et al.*; both for $\gamma\to 0$ (but $\gamma\neq 0$) and for $\gamma\to\infty$. (b) The expectation value P of BW's observable for the even macroscopic quantum interference state is plotted against α for $\gamma=2$ (solid) and $\gamma=5$ (dashed). For $\gamma\gg 1$, the displacement operator acts like a rotation so that the parity of the even and odd macroscopic quantum interference states may be well flipped.

$|\langle e|D^\dagger(i\alpha_i)R_x(\theta)|e\rangle|^2=|\langle d|D^\dagger(i\alpha_i)R_x(\theta)|d\rangle|^2\to 1$ for $\gamma\to\infty$, where $\theta=2\gamma\alpha_i$ and α_i is real. As a result, the parity of the even and odd macroscopic quantum interference states, which are the orthogonal entangled elements in the entangled coherent state, can be perfectly flipped by the displacement operator for $\gamma\to\infty$ as is implied in Fig. 2(b) [10]. This property enables the maximal Bell violation of the entangled coherent state for a large coherent amplitude.

In the pseudospin formalism, the correlation function $E(\theta_1,\varphi_1,\theta_2,\varphi_2)=\langle \mathrm{ECS}|s_1(\theta_1,\varphi_1)\otimes s_2(\theta_2,\varphi_2)|\mathrm{ECS}\rangle$ of the entangled coherent state is

$$E(\theta_1,\varphi_1,\theta_2,\varphi_2)=-\cos\theta_1\cos\theta_2-K(\gamma)\times\cos(\varphi_1-\varphi_2)\sin\theta_1\sin\theta_2,$$

$$K(\gamma)=\frac{\cosh\gamma^2\sinh\gamma^2}{\left(\sum_{n=0}^{\infty}\frac{\gamma^{4n+1}}{\sqrt{(2n)!(2n+1)!}}\right)^2}, \tag{20}$$

where $0<K(\gamma)<1$, and $K(\gamma)$ approaches 1 when $\gamma\to 0$ (but $\gamma\neq 0$) and $\gamma\to\infty$. The maximized value of the Bell function $B=\langle\mathcal{B}\rangle$ is obtained from Eq. (20) as

$$|B|_{max}=2\sqrt{1+K(\gamma)^2}, \tag{21}$$

by setting $\theta_1=0$, $\theta_1'=\pi/2$, $\theta_2=-\theta_2'$, and $\varphi_1=\varphi_2=0$. Then, the maximal violation is found for the two extreme cases, $\gamma\to 0$ and $\gamma\to\infty$. When γ is small, the entangled coherent state is not maximally entangled in an infinite-dimensional Hilbert space as tracing the state over one mode variables the von Neumann entropy is not infinite. It is interesting to note that the nonmaximally entangled state maximally violates the Bell's inequality. We attribute this mismatch to the dichotomic nature of the test of quantum nonlocality for an infinite-dimensional system. However, the entangled coherent state is maximally entangled in the 2×2-Hilbert space, but it does not always maximally violate the Bell-CHSH inequality as shown in Fig. 2(a). This shows that the pseudospin formalism is not a "perfect" analogy of a two-qubit system when a qubit is composed of two orthogonal even and odd macroscopic quantum interference states. The pseudospin operator $\mathbf{a}\cdot\mathbf{s}$ (with $\varphi=0$) in Eq. (3) can be written as $\mathbf{a}\cdot\mathbf{s}=U(\theta)s_z$, where a unitary rotation $U(\theta)$ is

$$U(\theta)|2n+1\rangle=\cos\theta|2n+1\rangle+\sin\theta|2n\rangle, \tag{22}$$

$$U(\theta)|2n\rangle=-\sin\theta|2n+1\rangle+\cos\theta|2n\rangle. \tag{23}$$

The even (odd) macroscopic quantum interference state does not flip into the odd (even) macroscopic quantum interference state by $U(\theta)$; it is only the parity of the given state which changes. The fidelity between the "rotated" odd macroscopic quantum interference state and the even macroscopic quantum interference state is obtained as

$$|\langle d|U(\pi/2)|e\rangle|^2=K(\gamma), \tag{24}$$

which is smaller than 1. It is clear that $|e\rangle$ and $|d\rangle$ are well flipped to each other only for the limiting cases of $\gamma\to 0$ and $\gamma\to\infty$. In other word, the rotation may get the given states out of the 2×2 space spanned by $|e\rangle$ and $|d\rangle$. Note, for example, that $U(\pi/2)|e\rangle$ cannot be represented by a linear superposition of $|e\rangle$ and $|d\rangle$.

IV. THE CLAUSER-HORNE INEQUALITY

We have studied quantum nonlocality of continuous-variable states using the Bell-CHSH inequality [3] and all the arguments have been based upon the parity measurement. The Clauser and Horne's version of Bell's inequality [4] can also be considered to test the nonlocality of continuous-variable states with photon number measurement [6]. We will investigate the Bell-CH inequality in this section.

A. The bound values for Bell-CH inequality

The bound values for the Bell-CHSH inequality $\pm 2\sqrt{2}$ are well known as Cirel'son bound [12]. The upper bound $(-1+\sqrt{2})/2$ of the Bell-CH inequality was proved by comparing the CH and CHSH inequalities [21]. The bound values for the Bell-CH inequality can also be simply found as follows. The Bell-CH operator for a two-qubit system is defined as [4,6]

$$\mathcal{B}_{CH}=\xi_1(\theta_1)\otimes\xi_2(\theta_2)+\xi_1(\theta_1)\otimes\xi_2(\theta_2')+\xi_1(\theta_1')\otimes\xi_2(\theta_2)-\xi_1(\theta_1')\otimes\xi_2(\theta_2')-\xi_1(\theta_1)\otimes\mathbb{1}_2-\mathbb{1}_1\otimes\xi_2(\theta_2), \quad (25)$$

where

$$\xi(\theta)=|\theta\rangle\langle\theta|, \quad (26)$$

$$|\theta\rangle=\cos\theta|0\rangle+\sin\theta|1\rangle, \quad (27)$$

then the local theory imposes the inequality $-1\leqslant\langle\mathcal{B}_{CH}\rangle\leqslant 0$. Note here that we investigate a simple 2×2 system without loss of generality. One can prove by direct calculation

$$\mathcal{B}_{CH}^2=-\mathcal{B}_{CH}-\Delta, \quad (28)$$

where

$$\Delta=\langle\theta_1|\theta_1'\rangle(|\theta_1\rangle\langle\theta_1'|-|\theta_1'\rangle\langle\theta_1|)_1\otimes\langle\theta_2|\theta_2'\rangle(|\theta_2\rangle\langle\theta_2'|-|\theta_2'\rangle\langle\theta_2|)_2. \quad (29)$$

Using $\langle\mathcal{B}_{CH}\rangle^2\leqslant\langle\mathcal{B}_{CH}^2\rangle$, the average of Eq. (28) becomes

$$\langle\mathcal{B}_{CH}\rangle^2+\langle\mathcal{B}_{CH}\rangle+\langle\Delta\rangle\leqslant 0, \quad (30)$$

and the Bell-CH function $B_{CH}\equiv\langle\mathcal{B}_{CH}\rangle$ is

$$\frac{-1-\sqrt{1-4\langle\Delta\rangle}}{2}\leqslant B_{CH}\leqslant\frac{-1+\sqrt{1-4\langle\Delta\rangle}}{2}. \quad (31)$$

The maximal and minimal values of $\langle\Delta\rangle$ can be obtained from the eigenvalues of Δ [20], which are $\pm\sin[2(\theta_1-\theta_1')]\sin[2(\theta_2-\theta_2')]/4$. The inequality $-1/4\leqslant\langle\Delta\rangle\leqslant 1/4$ is then obtained. Finally, the maximum and minimum of the Bell-CH function are found at $\langle\Delta\rangle=-1/4$ as

$$\frac{-1-\sqrt{2}}{2}\leqslant B_{CH}\leqslant\frac{-1+\sqrt{2}}{2}, \quad (32)$$

in which the upper and lower bounds of the Bell-CH function are given. For example, the Bell-CH function for a single-photon entangled state

$$|\psi\rangle=\frac{1}{\sqrt{2}}(|0\rangle|1\rangle-|1\rangle|0\rangle) \quad (33)$$

is calculated to be

$$B_{CH}=\frac{1}{4}\{\cos[2(\theta_1'-\theta_2')-\cos[2(\theta_1-\theta_2')-\cos[2(\theta_1'-\theta_2)-\cos[2(\theta_1-\theta_2)]-2\}. \quad (34)$$

This maximizes to $(\sqrt{2}-1)/2\simeq 0.21$ at $\theta_1=0$, $\theta_2=-3\pi/8$, $\theta_1'=\pi/4$, and $\theta_2'=-5\pi/8$ [22] and minimizes to $-(\sqrt{2}+1)/2\simeq-1.21$ at $\theta_1=0$, $\theta_1'=\pi/4$, and $\theta_2=-\theta_2'=\pi/8$.

B. Bell-CH inequalities for continuous variables

BW used the Q function for the test of the Bell-CH inequality violation of the simple single-photon entangled state (33) [6]. The Q function for a two-mode state ρ_{12} is defined as

$$Q_{12}(\alpha,\beta)=\frac{{}_2\langle\beta|{}_1\langle\alpha|\rho_{12}|\alpha\rangle_1|\beta\rangle_2}{\pi^2}, \quad (35)$$

where $|\alpha\rangle$ and $|\beta\rangle$ are coherent states. The Bell-CH function in terms of Q representation is

$$\begin{aligned}B_{CH-BW}&=\langle\zeta_1(\alpha)\otimes\zeta_2(\beta)+\zeta_1(\alpha)\otimes\zeta_2(\beta')+\zeta_1(\alpha')\otimes\zeta_2(\beta)-\zeta_1(\alpha')\otimes\zeta_2(\beta')-\zeta_1(\alpha)\otimes\mathbb{1}_2-\mathbb{1}_1\otimes\zeta_2(\beta)\rangle\\&=\pi^2[Q_{12}(\alpha,\beta)+Q_{12}(\alpha,\beta')+Q_{12}(\alpha',\beta)-Q_{12}(\alpha',\beta')]-\pi[Q_1(\alpha)+Q_2(\beta)],\end{aligned} \quad (36)$$

where $Q_1(\alpha)$ and $Q_2(\beta)$ are the marginal Q functions of modes 1 and 2, and $\zeta(\alpha)=D(\alpha)|0\rangle\langle 0|D^{\dagger}(\alpha)$. Equation (36) is a generalized version of the BW's formalism as BW considered $\alpha=\beta=0$ [6]. In this case the measurement results are distinguished according to the presence of photons, in other words, the dichotomic outcomes are no photon and the presence of photons. This is more realistic because the parity of photon numbers is difficult to measure with currently developed photodetectors.

The Q function for the two-mode squeezed state is [15]

$$Q_{TMSS}(\alpha,\beta)=\frac{1}{\pi^2\cosh^2 r}\exp[-|\alpha|^2-|\beta|^2+\tanh r(\alpha\beta+\alpha^*\beta^*)], \quad (37)$$

and the Q function for the entangled coherent state

$$\begin{aligned}Q_{ECS}(\alpha,\beta)=\mathcal{N}^2\{&\exp[-|\alpha-\gamma|^2-|\beta+\gamma|^2]+\exp[-|\alpha+\gamma|^2-|\beta-\gamma|^2]-\exp[-(\alpha-\gamma)(\alpha^*+\gamma)\\&-(\beta+\gamma)(\beta^*-\gamma)-4\gamma^2]-\exp[-(\alpha^*-\gamma)\times(\alpha+\gamma)-(\beta^*+\gamma)(\beta-\gamma)-4\gamma^2]\}.\end{aligned} \quad (38)$$

The marginal Q function of each state can also be simply obtained from Eqs. (37) and (38). One can investigate the violation of the Bell-CH inequality for the two different states from Eqs. (36), (37), and (38). The results are plotted in Figs. 3(a) and 3(b).

For the two-mode squeezed state, the degree of the violation of the Bell-CH inequality increases as generalizing the BW formalism. However, it increases up to a peak and decreases as increasing the squeezing r, which is shown in Fig. 3(a). The two-mode squeezed state is a separable pure state when r is zero, where no violation of Bell's inequality is found. As r increases, entanglement becomes to exist, which causes the violation of Bell's inequality. However, as r increases, the average photon number increases and the weight

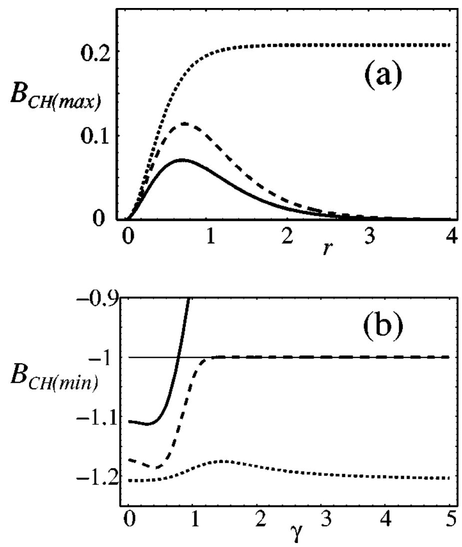

FIG. 3. (a) The maximized Bell-CH function $B_{CH(max)}$ for a two-mode squeezed state is plotted against the degree of squeezing r using the BW (solid line) and the generalized BW (dashed) formalisms. The maximized function $B_{CH(max)}$ of the same state based upon parity measurement for the same state is given (dotted line). (b) The minimized Bell-CH function $B_{CH(min)}$ for an entangled coherent state is plotted against its coherent amplitude γ using the BW (solid line) and the generalized BW (dashed) formalisms. The minimized function $B_{CH(min)}$ based upon the parity measurement is plotted for the same state (dotted line).

of $|0\rangle|0\rangle$ decreases as seen in Eq. (5). As the BW formalism of the Bell-CH violation is based on the nonlocality of no photon and presence of photons, its violation diminishes when r is large.

The even and odd cat states become the no-photon and single-photon number states respectively, i.e., $|e\rangle \rightarrow |0\rangle$ and $|d\rangle \rightarrow |1\rangle$, when $\gamma \rightarrow 0$. Therefore, the entangled coherent state approaches to the single-photon entangled state (33) in this limit. It can be simply shown that the degree of the Bell-CH violation in the generalized BW formalism for the entangled coherent state for $\gamma \rightarrow 0$ ($B_{CH-BW} \simeq -1.17$) is the same as that for single-photon entangled state (33). It is larger than the maximized value found by BW ($B_{CH-BW} \simeq -1.11$) [6] which is also shown in Fig. 3(b). However, it does not still reach the maximal violation $-(1+\sqrt{2})/2 \simeq -1.21$, in which the single-photon entangled state (33) shows with perfect rotations. It does not maximally violate the Bell-CH inequality because of the imperfect rotations by the displacement operator used in the BW formalism (36). Note that the displacement operator does not flip $|0\rangle$ to $|1\rangle$ and vice versa [see Fig. 1(b)]. As γ becomes large, one can observe qualitatively the same phenomenon as for the two-mode squeezed state. The Bell violation approaches zero as $\gamma \rightarrow \infty$ because of the decrease of the weight of the term $|0\rangle|0\rangle$.

Instead of the measurement of the presence of photons, the parity measurement can be used with the unitary rotation $U(\theta)$ to investigate the Bell-CH inequality. The Bell-CH function is defined as

$$B_{CH}^{(\mathrm{II})} = \langle \chi_1(\theta_1)\otimes\chi_2(\theta_2) + \chi_1(\theta_1)\otimes\chi_2(\theta_2') + \chi_1(\theta_1') \otimes\chi_2(\theta_2) - \chi_1(\theta_1')\otimes\chi_2(\theta_2') - \chi_1(\theta_1)\otimes\mathbb{1}_2 - \mathbb{1}_1 \otimes\chi_2(\theta_2)\rangle, \tag{39}$$

$$\chi(\theta) = \sum_{n=0}^{\infty} U(\theta)|2n\rangle\langle 2n|U^{\dagger}(\theta). \tag{40}$$

For the two-mode squeezed state,

$$\langle \chi_1(\theta_1)\otimes\chi_2(\theta_2)\rangle = \sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2\tanh 2r, \tag{41}$$

$$\langle \chi_1(\theta_1)\otimes\mathbb{1}\rangle = \frac{(\cos^2\theta_1\cosh^2 r + \sin^2\theta_1^2\sinh^2 r)}{\cosh 2r}, \tag{42}$$

and for the entangled coherent state,

$$\langle \chi_1(\theta_1)\otimes\chi_2(\theta_2)\rangle = \frac{1}{2}(\sin^2\theta_1\cos^2\theta_2 + \sin^2\theta_1\cos^2\theta_2) - K(\gamma)\sin\theta_1\cos\theta_1\sin\theta_2\cos\theta_2, \tag{43}$$

$$\langle \chi_1(\theta_1)\otimes\mathbb{1}\rangle = \frac{1}{2}(\cos^2\theta_1 + \sin^2\theta_1), \tag{44}$$

from which the Bell-CH function $B_{CH}^{(\mathrm{II})}$ can be obtained. In both cases, we find that the Bell-CH function approaches the maximal violation $B_{CH}^{(\mathrm{II})} \rightarrow -(1\pm\sqrt{2})/2$. For the two-mode squeezed state, $B_{CH}^{(\mathrm{II})}$ reaches the maximal violation for $r \rightarrow \infty$ as shown in Fig. 3(a). The upper bound is found at $\theta_1 = 0$, $\theta_2 = -3\pi/8$, $\theta_1' = \pi/4$, and $\theta_2' = -5\pi/8$, and the lower bound at $\theta_1 = 0$, $\theta_2 = -\theta_2' = \pi/8$, and $\theta_1' = \pi/4$. As shown in Fig. 3(b), for the entangled coherent state, $B_{CH}^{(\mathrm{II})}$ reaches the maximal violation for $\gamma \rightarrow 0$ and $\gamma \rightarrow \infty$ at the same angles.

V. REMARKS

We have studied the violation of Bell's inequalities using various formalisms. We have been able to discuss the link between the discussions for the quantum nonlocality of finite- and infinite-dimensional systems. The pseudospin operator [8] can be understood as the limiting case of Gisin-Peres observable [5]. The BW formalism [6] can be generalized to obtain a larger Bell violation [10]. However, the original EPR state cannot maximally violate Bell's inequality even in the generalized version of the BW formalism. We discussed the reason compared with the case of the entangled coherent state which shows the maximal violation of Bell's inequality in the generalized BW formalism. Our result is in agreement with the recent study of nonlocality of a two-mode squeezed state in absorbing optical fibers [23]. In Ref. [23], the authors found that nonlocality of the two-mode squeezed state is more robust against a dissipative environment in pseudospin approach than in the previous study [24] based on the BW formalism. It was shown that the dichotomic measurement for the presence of photons is not so effective in finding the nonlocality of two-mode squeezed states and entangled coherent states.

However, it must be pointed out that the nonlocality based on the Wigner and Q functions is extremely useful because we know the measurement of W and Q functions is experimentally possible while the implementation of other operations which we have discussed here have difficulties in their experimental realization.

Note added in proof. Recently, we have found that Banazek *et al.* studied the Bell-CH inequality for a single-photon entangled state and a two-mode squeezed state in terms of Q representation [25]. They took imperfect detection efficiency into consideration. Lately, a paper generalizing the work of Chen *et al.*, where different qubit states are assigned to a continuous variable system, has also appeared [26].

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